Differential Geometry I $\boxed{\text{Week 10}}$

We will now focus our attention on <u>surfaces</u> (i.e. 2-dimensional submanifolds) of $\mathbb{R}^3$. We will study their geometry by understanding the geometry of curves on them.

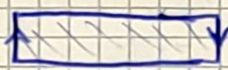
<u>Definition</u>: A <u>co-orientation</u> of a surface $S \subset \mathbb{R}^3$ of class C^1 is a continuous vector field $n : S \to \mathbb{R}^3$ such that $\|n\| = 1$ and $n(p) \perp T_pS \quad \forall p \in S$.

S is called <u>co-orientable</u> if it admits a co-orientation.

<u>Note</u> n unique up to $\pm$. If S connected and co-orientable: Two possible co-orientations.

<u>Remarks</u>: i) Locally every surface is co-orientable (the image of a local parametrization always is - see below)

ii) S of class $C^k \Rightarrow n$ of class at least C^{k-1}

iii) <u>Möbius strip</u>: Not co-orientable. Obtained by gluing the strip  at the two short edges in the direction indicated.

iv) <u>Orientation</u> of a surface: A continuous choice of orientation of $T_pS \ \forall p \in S$. Equivalent to a co-orientation.

<u>Examples</u>: • If $S = \{f(x,y,z) = 0\}$ for some f with $\nabla f \neq 0$, $f \in C^1$.

Then $n = \dfrac{\nabla f}{\|\nabla f\|}$ is a continuous unit normal, so S is co-orientable

• If S is globally parametrized by $\psi : \Omega \subseteq \mathbb{R}^2 \to S \subset \mathbb{R}^3$:

If (u,v) are coordinates on Ω, then $\forall p \in \Omega$, $\dfrac{\partial \psi}{\partial u}, \dfrac{\partial \psi}{\partial v}$ are a basis of $T_{\psi(p)}S$ and

$$n = \frac{\frac{\partial \psi}{\partial u} \times \frac{\partial \psi}{\partial v}}{\left\| \frac{\partial \psi}{\partial u} \times \frac{\partial \psi}{\partial v} \right\|}$$

is a continuous unit normal.

Def: Let $S \subset \mathbb{R}^3$ be a co-oriented surface. The Gauss map on S is the map $n: S \longrightarrow S^2$ (since $\|n\| = 1$, we have $n(p) \in S^2$ for every p)

- If S = flat plane: The Gauss map is constant
- The Gauss map captures the "curvature" of the surface.

Geodesics:

The analogues of "straight" curves on surfaces:

Def: A curve $\gamma: I \to S \subset \mathbb{R}^3$ of class C^2 is a geodesic of S if, $\forall t \in I$, $\ddot{\gamma}(t) \perp T_{\gamma(t)} S$.

If S is co-oriented: This is equivalent to $\ddot{\gamma}(t) \times n(\gamma(t)) = 0$.

Lem: If γ is a geodesic, then it has a constant speed $\|\dot{\gamma}\|$

Proof: Since γ is a curve on S, $\dot{\gamma}(t)$ is tangent to S at $\gamma(t)$.

So if γ is a geodesic $\Rightarrow \ddot{\gamma}(t) \perp \dot{\gamma}(t)$.

So $\frac{d}{dt}\langle \dot{\gamma}, \dot{\gamma} \rangle = 2\langle \ddot{\gamma}, \dot{\gamma} \rangle = 0 \Rightarrow \langle \dot{\gamma}, \dot{\gamma} \rangle = \text{const}$ $\square$

Simple examples:

- If S is a plane: Geodesics of S are straight lines (Since recall: If γ is a planar curve, $\ddot{\gamma}$ belongs in the plane, so if $\ddot{\gamma}$ is also perpendicular to that, it has to vanish)
- If S is the round sphere: Geodesics are great circles (parametrized with constant speed)

  (Exercise).

Darboux frame adapted to a surface:

Recall: If $\gamma: I \to \mathbb{R}^3$ is C^3 and biregular: Frenet frame $\{T_\gamma, N_\gamma, B_\gamma\}$, where $\{N_\gamma, B_\gamma\}$ span $T_\gamma^\perp$.

If γ lies in a surface S: I have another natural frame.

Def. Let $\gamma: I \longrightarrow S \subseteq \mathbb{R}^3$ be a C^2, regular curve on a co-oriented surface S.

i) The Darboux frame of γ: The orthonormal frame $\{T_\gamma, \nu_\gamma, n(\gamma(t))\}$, where $T_\gamma = \frac{\dot{\gamma}}{\|\dot{\gamma}\|}$ and $\nu_\gamma(t) = n(\gamma(t)) \times T_\gamma(t)$

Note: $\{T_\gamma(t), \nu_\gamma(t)\}$ is an orthonormal frame of $T_{\gamma(t)}S$

which is positively oriented with respect to the orientation of $T_{\gamma(t)}S$ associated to the co-orient. of S.

ii) If $K_\gamma(t)$ is the curvature vector of γ, then:

- Normal curvature: $k_n(t) = \langle K_\gamma(t), n(\gamma(t)) \rangle$
- Geodesic curvature: $k_g(t) = \langle K_\gamma(t), \nu_\gamma(t) \rangle$

Remarks: Since $\{n(t), \nu(t)\}$ span the space $T_\gamma^\perp$: The curvature vector can be written as

$K_\gamma(t) = k_n(t)\, n(t) + k_g(t) \cdot \nu(t)$, while the acceleration formula becomes

$$\frac{1}{v_\gamma^2}\ddot{\gamma}(t) = k_n(t)\, \nu(t) + k_g(t) \cdot \nu(t) + \frac{\dot{v}_\gamma(t)}{v_\gamma^2(t)} T_\gamma(t),$$ where $v_\gamma(t) = \|\dot{\gamma}(t)\|$.

So:

- γ is a geodesic of S iff it has constant speed and $k_g = 0$
  (In general: k_g measures the acceleration relative to geodesics)
- In general:

$$k_g^2(t) + k_n^2(t) = \|K_\gamma(t)\|^2 = \kappa^2(t) \quad \text{(scalar curvature of } \gamma)$$

<u>Def</u>: Geodesic torsion of γ: $\tau_g(t) = \frac{1}{\|\dot\gamma(t)\|} \langle \dot n(t), \mu(t) \rangle$

(Exercise: All the above quantities are invariant under reparametrization)

<u>Proposition</u>: Darboux equation

$$\begin{cases} \frac{1}{\|\dot\gamma(t)\|} \frac{d}{dt} T_\gamma = k_g \mu + k_n n \\ \frac{1}{\|\dot\gamma(t)\|} \frac{d}{dt} \mu = -k_g T_\gamma - \tau_g n \\ \frac{1}{\|\dot\gamma(t)\|} \frac{d}{dt} n = -k_n T_\gamma + \tau_g \mu \end{cases}$$

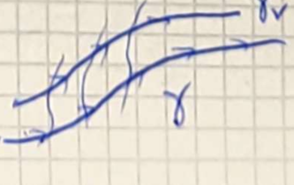
<u>Geodesics and length</u>

Like straight lines in a plane, geodesics arise as minimizers for the length functional.

To see this, we need to introduce the notion of a variation of a curve.

<u>Def</u>: Let $S \subset \mathbb{R}^3$ be a C^2 surface and $\gamma: I \to S$ be a C^2 regular curve. A <u>deformation</u> (or <u>variation</u>) of the curve γ is a C^2 map $\psi: I \times (-\varepsilon, \varepsilon) \to S$ such that $\psi(u, 0) = \gamma(u)$.

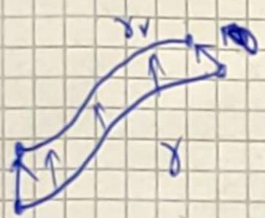
We denote $\gamma_v(u) = \psi(u, v)$ (so that $\gamma_0 = \gamma$) so that, for each $v \in (-\varepsilon, \varepsilon)$, $\gamma_v: I \to S$ is a C^2 curve.



Note: If I is compact, then for small values of v:
γ_v is also regular.

Theorem: (First variation formula for the length) Let $\gamma: [a,b] \to S$ be as above and assume, for the sake of simplicity, that it is naturally parametrized. Let $\psi: [a,b] \times (-\varepsilon, \varepsilon) \to S$ be a C^2 variation of γ.
Then the function $v \longrightarrow \ell(\gamma_v)$ satisfies

$$\frac{d}{dv} \ell(\gamma_v)\Big|_{v=0} = \left\langle \frac{\partial \psi}{\partial v}, \dot{\gamma} \right\rangle\Big|_{u=a}^{b} - \int_a^b k_g(u) \left\langle \frac{\partial \psi}{\partial v}, \nu(u) \right\rangle du$$



where $\frac{\partial \psi}{\partial v} = \frac{\partial \psi}{\partial v}(u,0)$ above.

Proof: $\ell(\gamma_v) = \int_a^b \sqrt{\langle \dot{\gamma}_v(u), \dot{\gamma}_v(u) \rangle}\, du = \int_a^b \sqrt{\left\langle \frac{\partial \psi}{\partial u}(u,v), \frac{\partial \psi}{\partial u}(u,v) \right\rangle}\, du$

So $$\frac{d}{dv}(\ell(\gamma_v))\Big|_{v=0} = \int_a^b \frac{\partial}{\partial v}\left(\sqrt{\left\langle \frac{\partial \psi}{\partial u}(u,v), \frac{\partial \psi}{\partial u}(u,v) \right\rangle}\right)\Big|_{v=0} du$$

$$= \int_a^b \frac{1}{\sqrt{\left\langle \frac{\partial \psi}{\partial u}(u,0), \frac{\partial \psi}{\partial u}(u,0) \right\rangle}} \cdot \left\langle \frac{\partial^2 \psi}{\partial u \partial v}(u,0), \frac{\partial \psi}{\partial u}(u,0) \right\rangle du.$$

Since $\gamma = \gamma_0$ is naturally parametrized: $\left\langle \frac{\partial \psi}{\partial u}(u,0), \frac{\partial \psi}{\partial u}(u,0) \right\rangle = \langle \dot{\gamma}, \dot{\gamma} \rangle = 1$

So
$$\frac{d}{dv}(\ell(\gamma_v))\Big|_{v=0} = \int_a^b \left\langle \frac{\partial}{\partial u} \frac{\partial \psi}{\partial v}(u,0), \frac{\partial \psi}{\partial u}(u,0) \right\rangle du \overset{\text{(Integrate by parts)}}{=}$$

$$= \left\langle \frac{\partial \psi}{\partial v}(u,0), \underbrace{\frac{\partial \psi}{\partial u}(u,0)}_{\dot{\gamma}(u)} \right\rangle\Big|_{u=a}^{b} - \int_a^b \left\langle \frac{\partial \psi}{\partial v}(u,0), \underbrace{\frac{\partial^2 \psi}{\partial u^2}(u,0)}_{\ddot{\gamma}(u)} \right\rangle du$$

But $\ddot{\gamma}(u) = K_\gamma(u) = k_n(u) \cdot N(u) + k_g(u) \cdot \nu(u)$

Since $\frac{\partial \psi}{\partial v}$ is tangent to S (and therefore perpendicular to $\vec{\nu}$)

We get $\left\langle \frac{\partial \psi}{\partial v}(u,0), K_\gamma(u) \right\rangle = \left\langle \frac{\partial \psi}{\partial v}(u,0), k_g(u)\cdot \mu(u) \right\rangle$. $\boxed{//}$

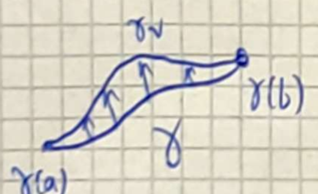
So if we choose variations that fix the endpoints:

<u>Corollary</u>: Let $\gamma : [a,b] \to S$ be of class C^2 and of constant speed. If γ minimizes the length among all curves connecting $\gamma(a)$ to $\gamma(b)$ on S, then γ is a geodesic.

<u>Remark</u>: • In this case, $\ell(\gamma) = d_S(\gamma(a), \gamma(b))$.

• The converse not true (consider a full great circle on the sphere).

<u>Proof</u>: Assume that γ as above and that, without loss of generality, $\|\dot\gamma(t)\| = 1$. Let $\psi : [a,b] \times (-\varepsilon, \varepsilon) \to S$ be a variation of γ fixing the endpoints, i.e. $\psi(a,v) = \gamma(a)$ and $\psi(b,v) = \gamma(b)$ for all $v \in (-\varepsilon, \varepsilon)$. Then, by assumption,



$\ell(\gamma_v)$ has a minimum at $v = 0$

$$\Rightarrow \frac{d}{dv} \ell(\gamma_v)\Big|_{v=0} = 0.$$

By the first variation formula, since $\frac{\partial \psi}{\partial v}(a,0) = 0$ and $\frac{\partial \psi}{\partial v}(b,0) = 0$,

We get $$\int_a^b k_g(u) \left\langle \frac{\partial \psi}{\partial v}(u,0), \mu(u) \right\rangle du = 0.$$

The above is true for any such variation. Note that, for any $\xi : [a,b] \to \mathbb{R}^3$ such that $\xi(u) \in T_{\gamma(u)} S$, and $\xi(a) = \xi(b) = 0$,

I can find a variation ψ fixing the endpoints such that $\frac{\partial \psi}{\partial v}(u,0) = \xi(u)$. So, choosing $\xi(u) = \chi(u) \cdot k_g(u) \cdot \psi(u)$

for some smooth function $\chi : [a,b] \to [0,+\infty)$ such that $\chi(a) = \chi(b) = 0$ but $\chi(u) > 0$ for $u \in (a,b)$, I get

$$\int_a^b k_g^2(u) \cdot \chi(u) \cdot \underbrace{\|\psi(u)\|^2}_{=1} \, du = 0 \implies k_g(u) = 0 \quad \forall u \in [a,b]$$

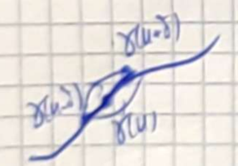
So γ is a geodesic

While the converse is not true, it is true in a local sense:

<u>Theorem</u>: A C^2 curve $\gamma : I \to S$ is a geodesic if and only if:

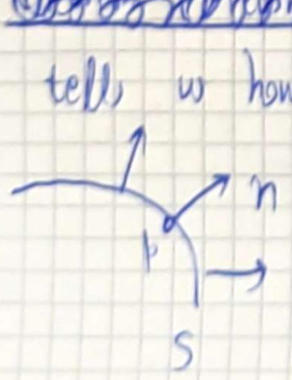
i) it has constant speed and

ii) it minimizes length locally, i.e. $\forall u \in I$, $\exists \delta > 0$ such that $\gamma|_{[u-\delta, u+\delta]}$ minimizes the length among curves on S connecting $\gamma(u-\delta)$ to $\gamma(u+\delta)$.



Back to the Gauss map: $n : S \to \mathbb{S}^2 \subset \mathbb{R}^3$:

The differential $dn_p : T_p S \to T_{n(p)} \mathbb{S}^2 \subset \mathbb{R}^3$ tells us how S is "curved" around p



Note: If $\gamma : I \to S$ is a curve in S, then

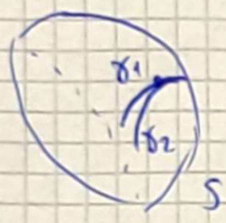
$$dn_{\gamma(t)}(\dot{\gamma}(t)) = \frac{d}{dt}(n \circ \gamma(t))$$

Theorem (Meusnier)

Let $\gamma: I \to S$ be a curve of class C^2. At any point $\gamma(t)$, the normal curvature depends only on the direction $\dot{\gamma}(t)$ (i.e. not on $\ddot{\gamma}(t)$), with

$$k_n(t) = -\frac{\langle dn(\dot{\gamma}(t)), \dot{\gamma}(t)\rangle}{\|\dot{\gamma}(t)\|^2} = -\langle dn(T_\gamma), T_\gamma\rangle$$

Remark: So k_n captures the curvature of the surface in the direction of $\dot{\gamma}$.



So: If γ_1, γ_2 two curves on S with $\gamma_1(0) = \gamma_2(0)$ and $\dot{\gamma}_1(0) \parallel \dot{\gamma}_1(0)$:

$$(k_n)_1(0) = (k_n)_2(0)$$

Recall also for a curve γ: $k^2 = k_g^2 + k_n^2$

So: A geodesic through p is the curve of minimal scalar curvature going through p in that direction.

(In other words: k_n is the "minimal" acceleration for a curve to stay on S)

Proof: Since $n(\gamma(t)) \perp T_{\gamma(t)} S$: $\langle n(\gamma(t)), \dot{\gamma}(t)\rangle = 0$

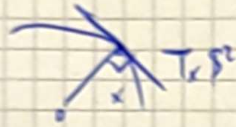
So, $0 = \frac{d}{dt}\langle n(\gamma(t)), \dot{\gamma}(t)\rangle = \langle \frac{d}{dt}(n \circ \gamma(t)), \dot{\gamma}(t)\rangle + \langle n(\gamma(t)), \ddot{\gamma}(t)\rangle$

$$\overset{\text{acceleration formula}}{=} \langle dn_{\gamma(t)}(\dot{\gamma}(t)), \dot{\gamma}(t)\rangle + \langle n(\gamma(t)), \|\dot{\gamma}(t)\|^2 k_n n + \|\dot{\gamma}\|^2 k_g \nu + \dot{v}_\gamma \cdot T_\gamma\rangle$$

But $\nu, T_\gamma \perp n$. $\blacksquare$

Definition: If $S \subset \mathbb{R}^3$ is co-oriented of class C^2, then **Weingarten map** at $p \in S$: $dn_p : T_pS \longrightarrow T_{n(p)}\mathbb{S}^2 \subset \mathbb{R}^3$ (or shape operator)

Note: Since on $\mathbb{S}^2$, if $x \in \mathbb{S}^2$ then $x \perp T_x\mathbb{S}^2$



then: $n(p)$: → normal to T_pS
→ normal to $T_{n(p)}\mathbb{S}^2$

So we can identify T_pS with $T_{n(p)}\mathbb{S}^2$

And think of $L_p = dn_p : T_pS \longrightarrow T_pS$ (endomorphism)

Definition: Second fundamental form: The bilinear map

$h_p : T_pS \times T_pS \longrightarrow \mathbb{R}$ given by

$$h_p(v,w) = -\langle L_p(v), w\rangle$$ ← Sign: Different conventions...

Metric tensor (just the Euclidean inner product)
for calculation in basis

In general: • Metric tensor captures the infinitesimal internal deformations associated to a parametrization of the metric

• Second fundamental form: Captures the "extrinsic" way in which the surface curves

(They are related by the Gauss-Codazzi equations)

Note Meusnier's thm: $$k_n(t) = -\frac{\langle L_p(\dot\gamma), \dot\gamma\rangle}{\langle \dot\gamma, \dot\gamma\rangle} = \frac{h(\dot\gamma,\dot\gamma)}{\langle \dot\gamma,\dot\gamma\rangle}$$

Proposition: If $\psi : \Omega \to S$ is a C^2 parametrization of the surface S and $b_i = \frac{\partial \psi}{\partial u_i}$, $i = 1,2$, then in the basis $\{b_1, b_2\}$, the second fundamental form has components

$$h_{ij}(u) = h_{\psi(u)}(b_i, b_j) = \langle n(\psi(u)), \frac{\partial^2 \psi}{\partial u_i \partial u_j} \rangle$$

(Corollary: h is symmetric: $h(\xi, \zeta) = h(\zeta, \xi)$.)

Proof: Since $b_1, b_2 \perp n$, we have

$$0 = \frac{\partial}{\partial u_j} \langle n, b_i \rangle = \langle \frac{\partial}{\partial u_j} n(\psi(u)), b_i(u) \rangle + \langle n(\psi(u)), \frac{\partial}{\partial u_j} b_i(u) \rangle$$

$$= \langle dn_{\psi(u)}(\underbrace{\frac{\partial \psi}{\partial u_j}}_{b_j}), b_i \rangle + \langle n, \frac{\partial}{\partial u_j} \frac{\partial}{\partial u_i} \psi(u) \rangle$$

$$\Rightarrow \underbrace{- \langle dn_{\psi(u)}(b_j), b_i \rangle}_{h_{\psi(u)}(b_j, b_i)} = \langle n, \frac{\partial^2 \psi}{\partial u_i \partial u_j} \rangle \qquad \square$$